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ON THE TRADEOFF BETWEEN

DRIFT AND VARIANCE

by

A. R. Washburn

January 1980

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A particle with fixed speed v that simultaneously wants to behave evasively and drift from one point to another in two dimensions has a conflict: If it drifts the maximum distance vt in a fixed time t, then it is forced to travel in an absolutely unevasive straight line. On the other hand, drift will not be maximal if the particle's motion is some sort of an evasive random walk. The purpose of this note is to report on an exploration of quantitative tradeoffs between these objectives.

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1. INTRODUCTION

      A particle with fixed speed v that simultaneously wants to behave evasively and drift from one point to another in two dimensions has a conflict: If it drifts the maximum distance vt in a fixed time t, then it is forced to travel in an absolutely unevasive straight line. On the other hand, drift will not be maximal if the particle's motion is some sort of an evasive random walk. The purpose of this note is to report on an exploration of quantitative tradeoffs between these objectives.

2. PROBLEM FORMULATION

      Let the positive x-axis represent the desired direction of drift, and suppose that the target goes from the origin to (X_t, Y_t) in time t. Let $x(t) = E(X_t)$ and $y(t) = E(Y_t)$, let S_t be the distance from $(x(t), y(t))$ to (X_t, Y_t), and let R_t be the distance from the origin to (X_t, Y_t). Let $\sigma_t^2 = E(S_t^2)$ be the measure of variance and $x^2(t)$ be the measure of drift. Then

(1)
$$\sigma_t^2 + x^2(t) \leq \sigma_t^2 + x^2(t) + y^2(t)$$

$$= E(X_t^2 + Y_t^2) = E(R_t^2) \leq (vt)^2$$

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From (1), $(vt)^2$ is an upper bound on $\sigma_t^2 + x^2(t)$, and this upper bound will be nearly achieved if

a) $y(t) = 0$

b) the target's track is nearly a straight line,
since in that case $E(R_t^2) \underset{\sim}{} (vt)^2$.

For example, if the particle were to flip a coin to decide whether its course should be θ or $-\theta$, then $\sigma_t^2 + x^2(t) = (vt)^2$, with $x^2(t)$ being largest when θ is 0 and σ_t^2 being largest when θ is $\pi/2$. The course should never be changed at any time; if there is any possibility of a course change in $[0,t]$, then $\sigma_t^2 + x^2(t) < (vt)^2$.

The above analysis leaves one with a feeling of dissatisfaction with the measure of variance σ_t^2, since maximization of σ_t^2 with a constraint on $x^2(t)$ leads to the adoption of an intuitively unevasive motion. A tracker who saw the particle begin its motion would have no difficulty extrapolating the track if he ever lost contact; once the initial direction θ is known, the particle's motion is deterministic. If the time origin were taken to be any time greater than 0, and if the prediction of future position were the conditional expectation given all past movements, then the particle's motion would not be evasive at all.

The above considerations lead to the adoption of s_t^2 as the measure of variance, where s_t^2 is variance from the conditional expectation of position given all past movements, averaged over all past movements. In order to simplify computation of s_t^2,

assume that the particle's course is a stationary Markov stochastic process, in which case the predictive power of all past motion is the same as the predictive power of current course. There are many such processes, from which we select a discontinuous one and a continuous one for further study. The natural discontinuous process is a "random tour" [3], where the particle changes direction only at the jump points of a Poisson process. The natural continuous process is the Ornstein-Uhlenbeck process, which is the only stationary Markov process that is normal. Figure 1 shows the ratio $s_t^2/((vt)^2 - x^2(t))$ as a function of $(x(t)/vt)^2$, where s_t^2 is in both cases maximal for the given value of $x(t)$. The rest of this paper consists of the computations lying behind Figure 1. Note that in both cases

$$(2) \qquad s_t^2 \leq .381[(vt)^2 - x^2(t)] \qquad \text{(O-U, random tour)}$$

An additional result is relevant. In [2], Grenander formulates an analytic expression for s_t^2 when the particle's course is any stationary process. The expression is in general very cumbersome, but in the special case where the process is normal and $x(t) \approx vt$, Grenander is able to exhibit the stationary process (it is not Markov) that maximizes s_t^2. The maximum s_t^2 is

$$(3) \qquad s_t^2 = \frac{4}{\pi^2} ((vt)^2 - x^2(t)) = .405((vt)^2 - x^2(t)).$$

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Evidently, the natural way to discuss the tradeoff between drift and variance is in terms of the ratio $R_t = s_t^2/((vt)^2 - x^2(t))$. The maximum possible value of R_t amongst all stationary processes is unknown, but it appears to be considerably smaller than 1.0, which is the bound obtained from equation (1) by observing that $s_t \leq \sigma_t$.

3. RANDOM TOUR CALCULATIONS

Assume without loss of generality that the particle's speed is $v = 1.0$. The particle is assumed to pick an independent, identically distributed sequence of courses $\theta_1, \theta_2, \ldots$ from some distribution for which $E(\sin \theta) = 0$. Each course holds for a time that is exponential with mean $1/\lambda$, after which a new course is adopted, etc. Let $E(\cos \theta) = c_1$, $E(\cos^2 \theta) = c_2$, $x_\theta(t) = E(X_t | \theta_1 = \theta)$, $y_\theta(t) = E(Y_t | \theta_1 = \theta)$, $v_X(t) = E(X_t^2)$, $v_Y(t) = E(Y_t^2)$ and retain the definitions of $x(t)$ and $y(t)$ made earlier. Since

$$(4) \quad s_t^2 = E([X_t - x_\theta(t)]^2 + [Y_t - y_\theta(t)]^2)$$

$$= v_X(t) + v_Y(t) - E(x_\theta^2(t) + y_\theta^2(t)) ,$$

the functions that need to be determined are $x_\theta(t)$, $y_\theta(t)$, $v_X(t)$, and $v_Y(t)$.

We know

$$(5) \qquad E(x_\theta(t)) = x(t) = tc_1$$

Let U be the time of the first course change, and let $f(u) = \lambda \exp(-\lambda u)$ be the density function of U. Then

$$(6) \quad E(X_t | \theta_1 = \theta, U=u) = \begin{cases} u \cos \theta + x(t-u) & \text{if } u \leq t \\ \\ t \cos \theta & \text{if } u \geq t \end{cases}$$

Therefore, by conditional probability,

$$(7) \quad x_\theta(t) = \int_0^t [u \cos \theta + x(t-u)] \, f(u) \, du + t \cos \theta \int_t^\infty f(u) \, du$$

After performing the integrations,

$$(8) \quad x_\theta(t) = \cos \theta (1-(1+\lambda t) \exp(-\lambda t))/\lambda + x*f(t) + t \cos \theta \exp(-\lambda t),$$

where $x*f(t)$ is the convolution of $x(t)$ and $f(t)$. Let $X(s)$, $X_\theta(s)$, and $F(s)$ be the Laplace transforms of $x(t)$, $x_\theta(t)$, and $f(t)$, respectively. Then $X(s) = c_1/s^2$ and $F(s) = \lambda/(\lambda+s)$. After cancelling $t \cos \theta \exp(-\lambda t)$ in (8) and taking Laplace transforms of both sides,

$$(9) \quad \lambda X_\theta(s) = \cos \theta (1/s - 1/(\lambda+s)) + c_1 \lambda^2/(s^2(s+\lambda)) \quad .$$

Inverting the Laplace transform $X_\theta(s)$, and letting $z = \lambda t$,

$$(10) \quad \lambda x_\theta(t) = \cos \theta (1 - \exp(-z)) + c_1(z - 1 + \exp(-z)),$$

or

$$(11) \quad \lambda x_\theta(t) = (\cos \theta - c_1)(1 - \exp(-z)) + c_1 z \quad .$$

Squaring both sides of (11) and taking expected values,

$$(12) \quad \lambda^2 E(x_\theta^2(t)) = (c_2 - c_1^2)(1 - \exp(-z))^2 + c_1^2 z^2$$

There are no cross product terms in (12) because $E(\cos \theta - c_1) = 0$. $c_2 - c_1^2$ is just the variance of $\cos \theta$. A similar analysis shows

(13) $\lambda^2 E(y_\theta^2(t)) = (s_2 - s_1^2)(1 - \exp(-z))^2 + s_1^2 z^2$,

where $s_1 = E(\sin \theta) = 0$ and $s_2 = E(\sin^2 \theta)$. Adding (12) and (13) and noting that $c_2 + s_2 = 1$,

(14) $\lambda^2 E(x_\theta^2(t) + y_\theta^2(t)) = (1-c_1^2)(1 - \exp(-z))^2 + c_1^2 z^2$

We use a similar technique to obtain formulas for $v_X(t)$ and $v_Y(t)$.

(15) $E(X_t^2 | U=u) = \begin{cases} u^2 c_2 + 2u c_1 x(t-u) + v_X(t-u) & \text{if } u \leq t \\ \\ t^2 c_2 & \text{if } u \geq t \end{cases}$

(16) $v_X(t) = \int_0^t [u^2 c_2 + 2u c_1 x(t-u) + v_X(t-u)] \, f(u) \, du$

$+ t^2 c_2 \int_t^\infty f(u) \, du$

After doing the integration, cancelling the $t^2 c_2 \exp(-\lambda t)$ term, taking Laplace transforms, and simplifying,

(17) $V_X(s) = 2c_2/(s^2(\lambda+s)) + 2c_1^2\lambda/(s^3(\lambda+s))$,

where $V_X(s)$ is the Laplace transform of $v_X(t)$. Inverting, with $z = \lambda t$,

(18) $\lambda^2 v_X(t) = 2(c_2 - c_1^2)(z - 1 + \exp(-z)) + z^2 c_1^2$

Similarly,

(19) $\lambda^2 v_Y(t) = 2(s_2 - s_1^2)(z - 1 + \exp(-z)) + z^2 s_1^2$

Substituting (14), (18), and (19) into (4),

(20) $\lambda^2 s_t^2 = (1 - c_1^2)(2z - 2(1 - \exp(-z)) - (1 - \exp(-z))^2)$,

or

(21) $$s_t^2 = t^2(1 - c_1^2)\ g(z)\ ,$$

where

(22) $$g(z) = (2z - 2(1 - \exp(-z)) - (1 - \exp(-z))^2)/z^2$$

The function $g(z)$ has a maximum at $z = 1.9$, and $g(1.9) = .381$. Since $t^2(1-c_1^2) = t^2-x^2(t)$, (21) is thus consistent with (2). Note that λ should be set to make the number of turns in time t be 1.9, on the average.

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4. ORNSTEIN-UHLENBECK CALCULATIONS

The O-U process θ_t is governed by two numbers α and β. The equilibrium distribution is normal with mean 0 and variance β; i.e., $\theta_t \sim N(0,\beta)$. From Feller [1],

$$(23) \qquad \theta_u \sim N(p\theta_v, \beta(1-p^2)) \qquad\qquad \text{for} \quad u \geq v$$

where $p = \exp(-\alpha(u-v))$. The parameter α is thus a smoothing constant, with small values of α corresponding to smooth processes. In the following, we will repeatedly use the fact that, if $\theta \sim N(\mu,\sigma)$, then $E(\cos \theta) = (\cos \mu) \exp(-\sigma^2/2)$ and $E(\sin \theta) = (\sin \mu) \exp(-\sigma^2/2)$.

The notation and plan are as in the random tour analysis; i.e. we plan to employ (4) in obtaining an equation for s_t^2. We first note that

$$(24) \qquad x_\theta(t) = E(\int_0^t \cos \theta_u \, du) = \int_0^t E(\cos \theta_u) \, du$$

Employing (23) with $v = 0$ and $\theta_v = \theta$,

$$(25) \qquad x_\theta(t) = \int_0^t \cos(p_u \theta) \exp(-\beta(1-p_u^2)/2) \, du \ ,$$

where $p_u = \exp(-\alpha u)$. Therefore, since

$$[\int_a^b (h(x) \, dx)]^2 = \int_a^b \int_a^b h(x) \, h(y) \, dx \, dy,$$

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(26) $\quad x_\theta^2(t) = \int_0^t \int_0^t \cos(p_u\theta) \cos(p_v\theta) \exp(-\beta(1-p_u^2)/2)$

$$\times \exp(-\beta(1-p_v)^2/2)\, dudv$$

If cos is replaced by sin in (26), the result is an expression for $y_\theta^2(t)$. Using the fact that $\cos(p_u\theta)\cos(p_v\theta)+\sin(p_u\theta)\sin(p_v\theta)$ $= \cos((p_u - p_v)\theta)$, we therefore have

(27) $\quad x_\theta^2(t) + y_\theta^2(t) = \int_0^t \int_0^t \cos((p_u-p_v)\theta) \exp(-\beta(2-p_u^2-p_v^2)/2)\, dudv.$

Since $(p_u-p_v)\theta \sim N(0, \beta(p_u-p_v)^2)$,

(28) $\quad E(x_\theta^2(t) + y_\theta^2(t))$

$$= \int_0^t \int_0^t \exp(-\beta(p_u-p_v)^2/2) \exp(-\beta(2-p_u^2-p_v^2)/2)\, dudv ,$$

or

(29) $\quad E(x_\theta^2(t) + y_\theta^2(t))$

$$= \int_0^t \int_0^t \exp(-\beta[1 - \exp(-\alpha(u+v))])\, dudv$$

We turn next to computation of $v_X(t)$ and $v_Y(t)$. Since $X(t) = \int_0^t \cos\theta_u\, du$,

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$$(30) \qquad X^2(t) = \int_0^t \int_0^t \cos\theta_u \cos\theta_v \, du \, dv.$$

If cos is replaced by sin in (30), an expression for $Y^2(t)$ results. Since $\cos\theta_u \cos\theta_v + \sin\theta_u \sin\theta_v = \cos(\theta_u - \theta_v)$,

$$(31) \qquad X^2(t) + Y^2(t) = \int_0^t \int_0^t \cos(\theta_u - \theta_v) \, du \, dv$$

From (23), $\theta_u - \theta_v \sim N(-\theta_v(1-p), \beta(1-p^2))$ for $u \geq\cdot v$. Since $\theta_v \sim N(0,\beta)$,

$$(32) \qquad \theta_u - \theta_v \sim N(0,\beta(1-p)^2 + \beta(1-p^2)) \qquad \text{for } u \geq v,$$

or

$$(33) \qquad (\theta_u - \theta_v) \sim N(0, \, 2\beta[1 - \exp(-\alpha|u-v|)])$$

Returning to (31), we finally obtain

$$(34) \quad E(X^2(t) + Y^2(t)) = \int_0^t \int_0^t \exp(-\beta[1 - \exp(-\alpha|u-v|)]) \, du \, dv$$

Substituting (34) and (29) into (4), one obtains a long but nonetheless explicit formula for s_t^2 as a function of α and β. Since

(35) $\qquad x(t) = tE(\cos \theta) = t \exp(-\beta/2)$,

maximizing s_t^2 for fixed $x(t)$ is the same as maximizing s_t^2 for fixed β. The maximized s_t^2, after being divided by $t^2 - x^2(t)$, is shown in Figure 1. Figure 2 shows the optimal product αt as a function of $(x(t)/t)^2$. Figure 3 shows s_t^2/t^2 as a function of αt for $\beta = 1$, showing that it is better for the particle to make αt too large than too small.

Further analysis is possible in case β is very large or very small. After making a change of variable for $|u-v|$ in (34) and for $u + v$ in (29), the result is, with $z = \alpha t$

(36) $\qquad s_t^2/t^2 = F(\beta ,z) \equiv \dfrac{4}{z^2} \int_0^z dx \int_0^x \Delta(y) dy$,

where

(37) $\qquad \Delta(y) = \exp[-\beta(1 - \exp(-y))] - \exp[-\beta(1 - \exp(-2y))]$

When β is very small,

(38) $\qquad\qquad \Delta(y) \underset{\sim}{} \beta[\exp(-y) - \exp(-2y)]$ $\qquad$ (small β)

After integrating (38) twice and multiplying by $(4/z^2)$, one obtains

(39) $\qquad\qquad F(\beta ,z) \underset{\sim}{} \beta g(z)$, $\qquad\qquad$ (small β)

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where $g(z)$ is the same function as in the random tour analysis (eqn (22)). Furthermore, since $t^2 - x^2(t) = t^2(1 - \exp(-\beta)) \approx t^2\beta$ when β is small,

$$(40) \qquad s_t^2/(t^2 - x^2(t)) \approx g(z) \qquad (\text{small } \beta)$$

When β is small, z should therefore be set to 1.9, in which case $g(z) = .381$.

Since $\lim_{\beta \to \infty} F(\beta, z) = 0$ for $z > 0$, the optimal z must approach 0 as β becomes large. Since $0 \le y \le x \le z$ in (36), y and x are small if z is. For small y,

$$(41) \qquad \Delta(y) \approx \exp(-\beta y) - \exp(-2\beta y) \qquad (\text{small } y)$$

After integrating (41) twice and multiplying by $(4/z^2)$, one obtains

$$(42) \qquad F(\beta, z) \approx g(\beta z) \qquad (\text{small } z)$$

where $g(\cdot)$ is once again the same function (22). Therefore

$$\lim_{\beta \to \infty} F(\beta, 1.9/\beta) = g(1.9) = .381;$$

that is, the particle can make s_t^2/t^2 asymptotically .381 when β is large by making $z = 1.9/\beta$. Since t^2 and $t^2 - x^2(t)$ are asymptotically equal when β is large, $s_t^2/(t^2 - x^2(t))$ is also asymptotically .381. Thus, the ratio $s_t^2/(t^2 - x^2(t))$ is bounded by .381 in all cases examined.

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REFERENCES

REFERENCES

1. Feller, William, "An Introduction to Probability Theory and its Applications," Vol. II, Wiley (p. 336).

2. Grenander, Ulf, "A Tactical Study of Evasive Maneuvers," FOA reports, Vol. 2, No. 4, Research Institute of National Reference Stockholm 80, Sweden (1968).

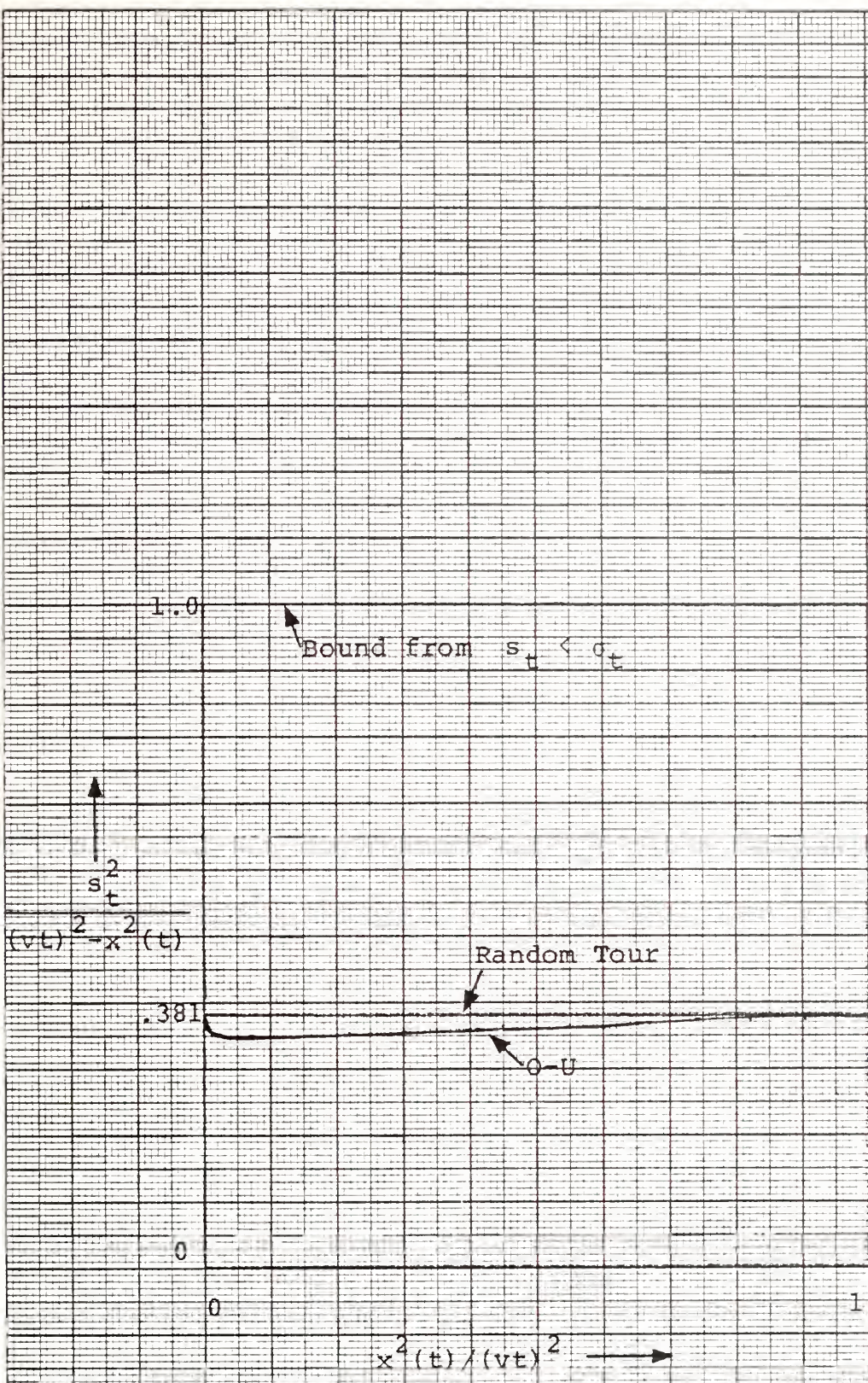3. Washburn, Alan, "Probability Density of a Moving Particle," Opns. Res. 17, 861-871 (1969).
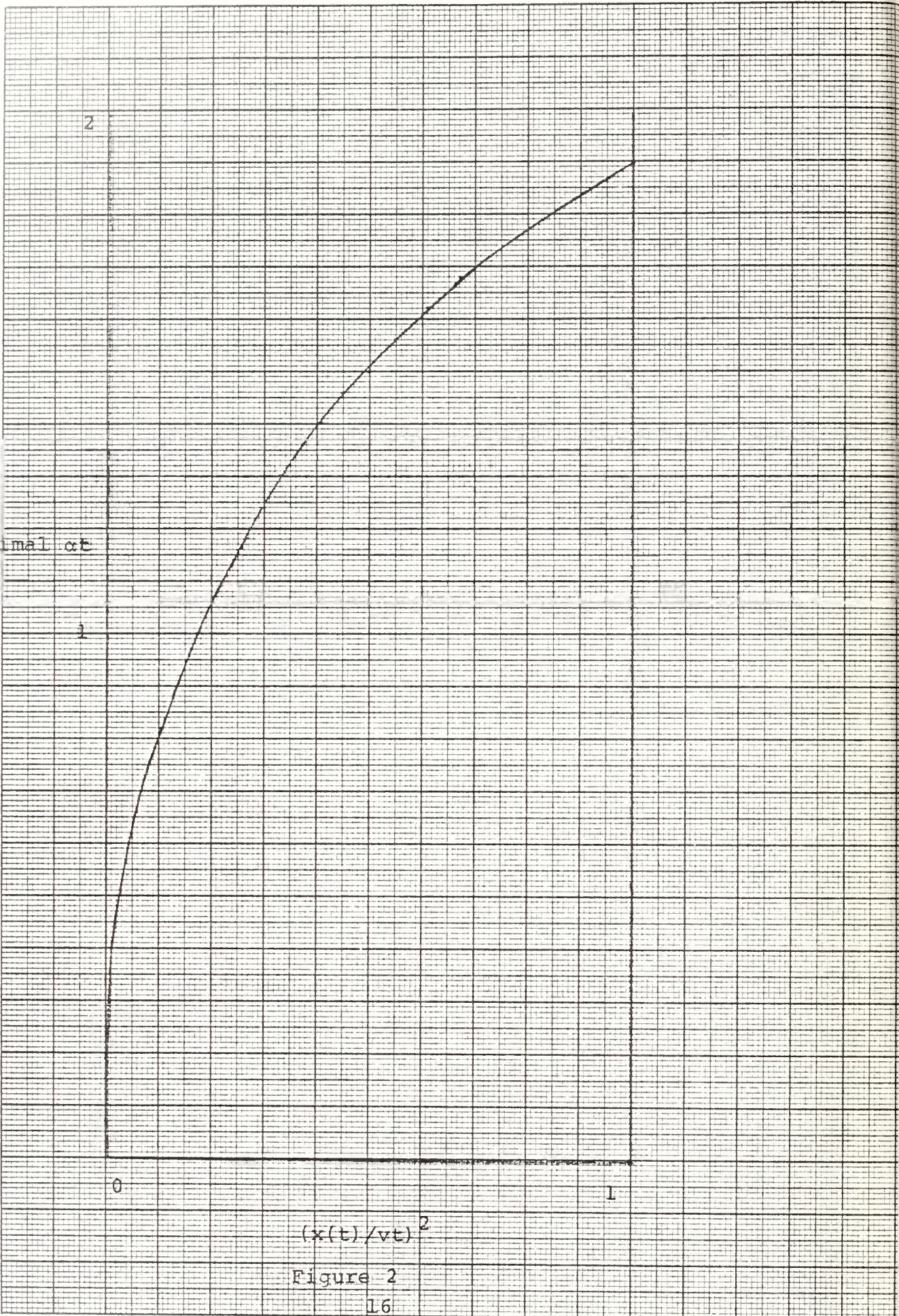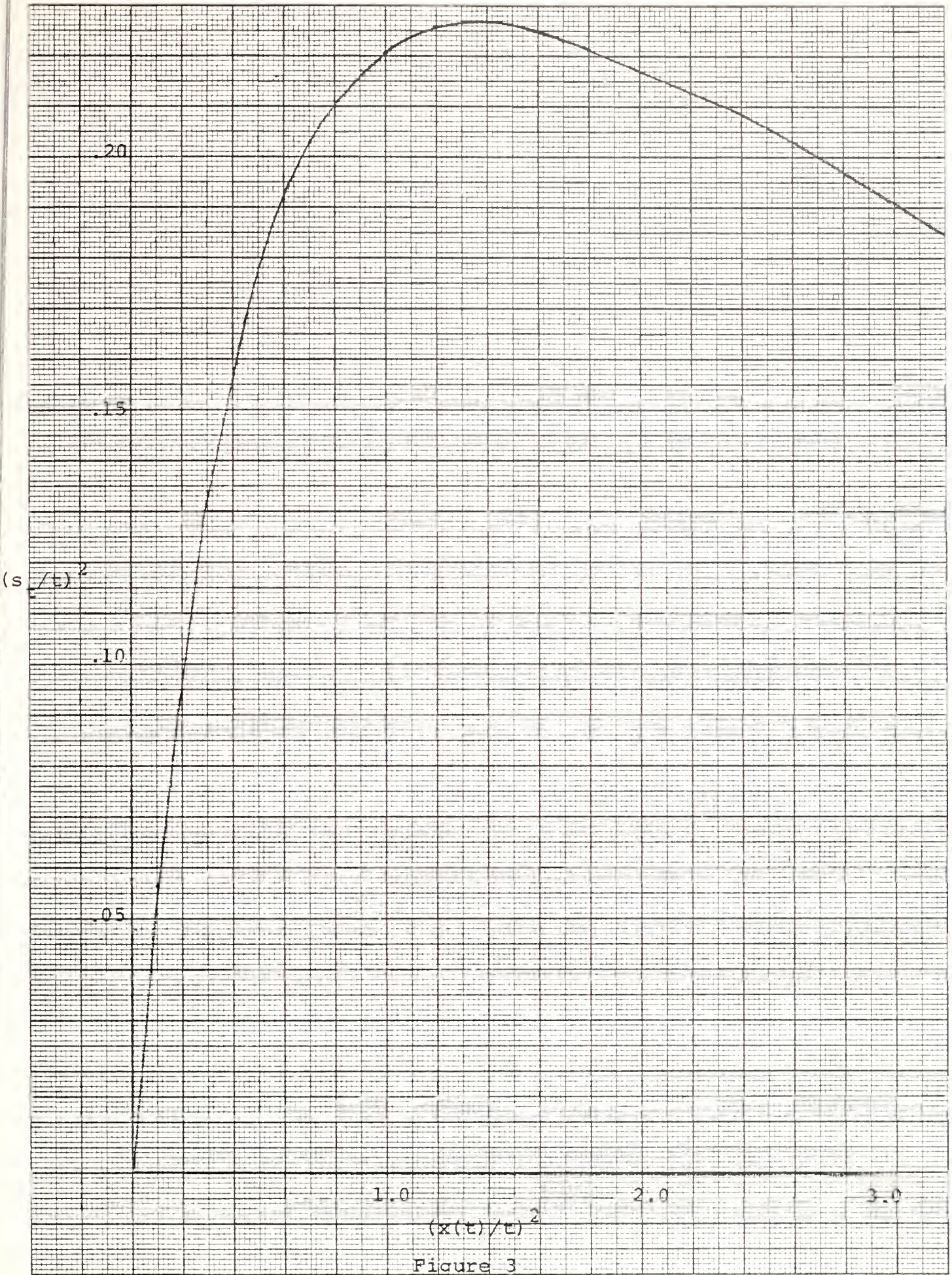
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Figure 2

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Figure 3

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